Wave Propagation in Junctions of Periodic Half-spaces

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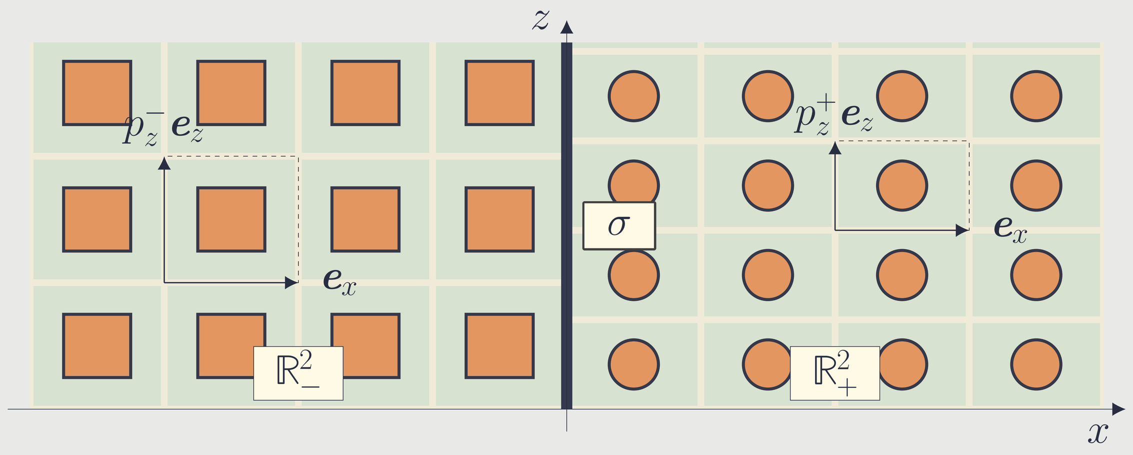
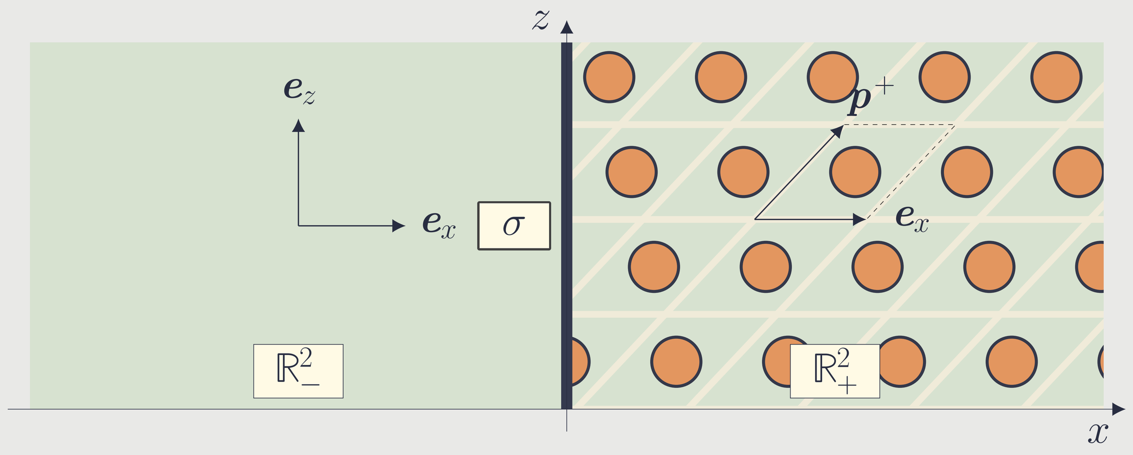
Model problem



$\mathscr{A}$ — $\mathbb{A}^\pm$ is $\mathbb{Z}\boldsymbol{e}_x + \mathbb{Z}(p_z^\pm \boldsymbol{e}_z)$–periodic
Let $\delta := p_z^+/p_z^-$

$$(\mathscr{P}) \quad \left| \begin{array}{l} \text{Find } u \in H^1(\mathbb{R}^2) \text{ such that} \\ -\,{}^t\nabla \mathbb{A}\, \nabla u - \rho\,\omega^2 u = 0 \quad \text{in} \quad \mathbb{R}^2_+ \cup \mathbb{R}^2_- \\ [\![\mathbb{A}\, \nabla u \cdot \boldsymbol{e}_x]\!]_\sigma = g \quad \text{on} \quad \sigma := \{0\} \times \mathbb{R}. \end{array} \right.$$

- $\mathbb{A}$ and ρ are bounded and coercive; $g \in L^2(\sigma)$
- $\mathfrak{Im}\,\omega > 0$ to ensure well-posedness in $H^1(\mathbb{R}^2)$
- $\mathbb{A}$ coincides with $\mathbb{A}^\pm$ on $\mathbb{R}^2_\pm$ (same for ρ).



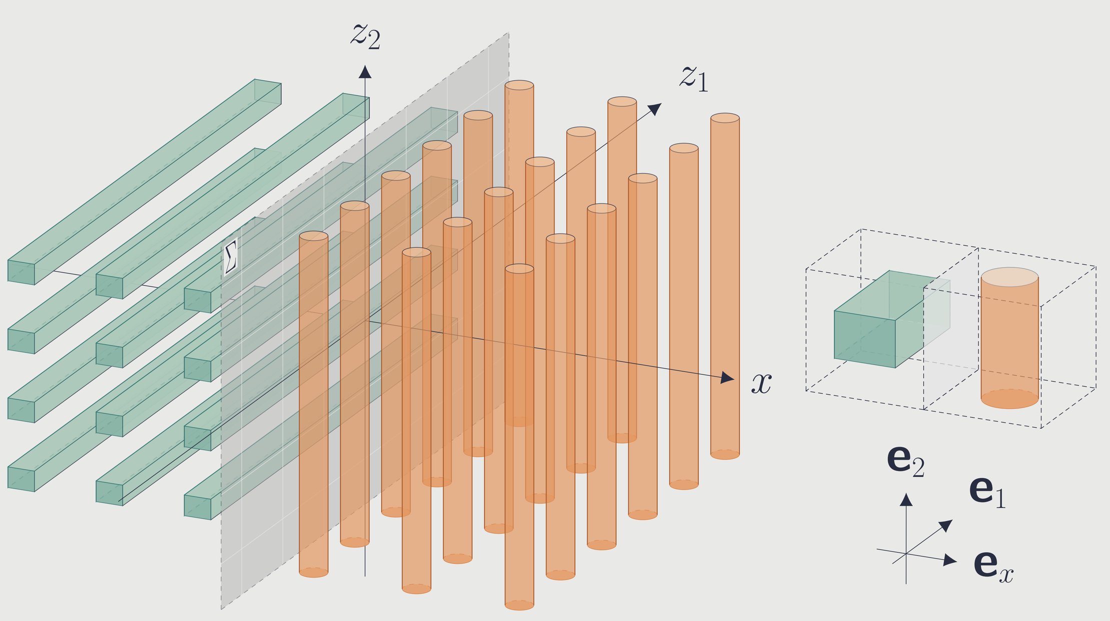
$\mathscr{B}$ — $\mathbb{A}^-$ is constant and $\mathbb{A}^+$ is $\mathbb{Z}\boldsymbol{e}_x + \mathbb{Z}\boldsymbol{p}^+$–periodic
Let $\delta := p_z^+$

1. If δ is rational then $\mathbb{A}$ and ρ are periodic in the direction of the interface σ.
   - Apply the Floquet-Bloch transform in the direction of σ and solve a family of waveguide problems set in $\mathbb{R} \times (0, \tau)$, τ being the period. [*Fliss, Joly*, 2009], [*Fliss, Cassan, Bernier*, 2010]
2. What if δ is irrational?

Goal: Solve $(\mathscr{P})$ in the irrational case.

Insight: Identify a so-called *quasiperiodic* structure.

The lifting approach



$$\mathscr{A} \quad \left| \begin{array}{l} \mathbb{A}_p^+(\mathbf{x}) = \mathbb{A}^+(x, z_1\, p_z^+) \\ \mathbb{A}_p^-(\mathbf{x}) = \mathbb{A}^-(x, z_2\, p_z^-) \end{array} \right. \quad \theta_1 = p_z^+, \quad \theta_2 = p_z^-.$$

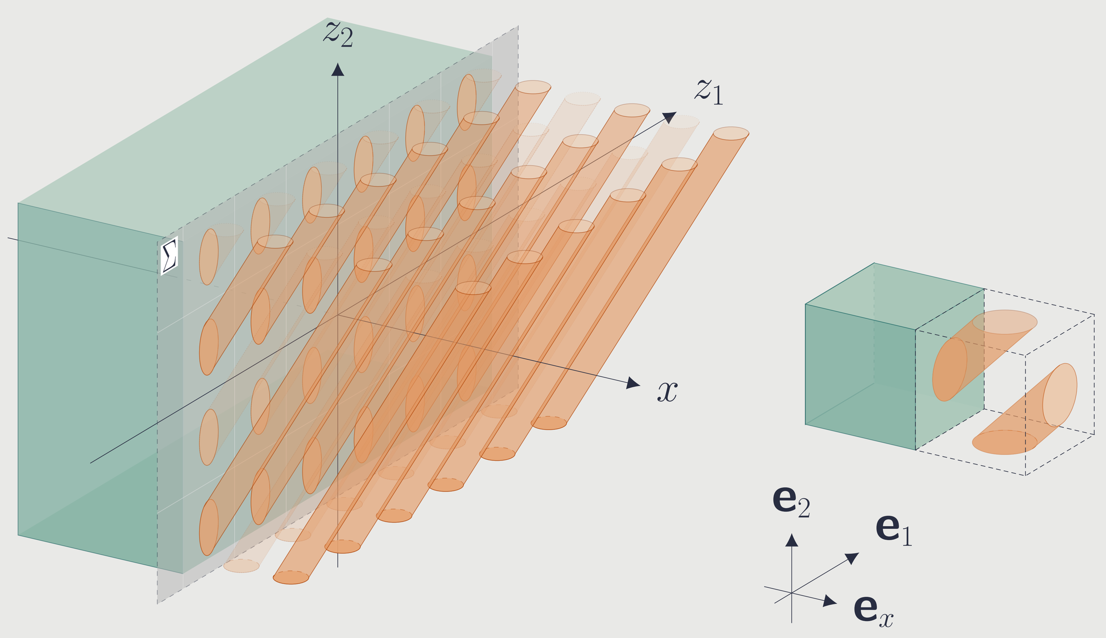
There exist 3D functions $\mathbb{A}_p^\pm$ and $\rho_p^\pm$ such that

- $\boxed{\mathbb{A}^\pm(\boldsymbol{x}) = \mathbb{A}_p^\pm(\Theta\, \boldsymbol{x}) \quad \text{and} \quad \rho^\pm(\boldsymbol{x}) = \rho_p^\pm(\Theta\, \boldsymbol{x})}$
- $\mathbb{A}_p^\pm$ and $\rho_p^\pm$ are $\mathbb{Z}^3$–periodic
- The *cut matrix* Θ is of the form $\Theta = \begin{pmatrix} 1 & 0 \\ 0 & \theta_1 \\ 0 & \theta_2 \end{pmatrix}$

Idea: Seek the solution u of $(\mathscr{P})$ under the form

$$\boxed{\text{a. e. } \boldsymbol{x} \in \mathbb{R}^2, \quad u(\boldsymbol{x}) = U(\Theta\, \boldsymbol{x})}$$

with U that satisfies a 3D transmission problem with periodicity along the interface.



$$\mathscr{B} \quad \left| \begin{array}{l} \mathbb{A}_p^+(\mathbf{x}) = \mathbb{A}^+(x + z_2\, p_x^+, z_1\, p_z^+) \\ \mathbb{A}_p^-(\mathbf{x}) = \mathbb{A}^- \end{array} \right. \quad \theta_1 = \frac{1}{p_z^+}, \quad \theta_2 = \frac{-p_x^-}{p_z^+}.$$

[*Gérard-Varet, Masmoudi*, 2011], [*Blanc, Le Bris, Lions*, 2015], [*Wellander, Guenneau, Cherkaev*, 2019]; Homogenization context.

The augmented problem

$$(\mathscr{P}\text{aug}) \quad \left| \begin{array}{l} \text{Find } U : \mathbb{R}^3 \to \mathbb{C} \text{ such that} \\ -\,{}^t\boldsymbol{\nabla}\, \Theta\, \mathbb{A}_p\, {}^t\Theta\, \boldsymbol{\nabla} U - \rho_p\,\omega^2\, U = 0 \quad \text{in} \quad \mathbb{R}^3_+ \cup \mathbb{R}^3_- \\ [\![(\Theta\, \mathbb{A}_p\, {}^t\Theta\, \boldsymbol{\nabla} U) \cdot \mathbf{e}_x]\!]_\Sigma = G \quad \text{on} \quad \Sigma := \{0\} \times \mathbb{R}^2. \end{array} \right.$$

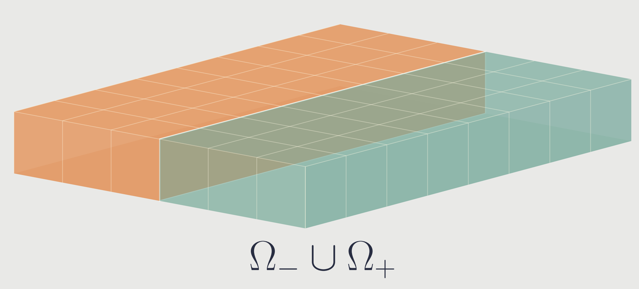
where
- $\mathbb{A}_p$ coincides with $\mathbb{A}_p^\pm$ on $\mathbb{R}^3_\pm$ (same for ρ_p)
- G is arbitrarily chosen such that $\boxed{G(\Theta\, \boldsymbol{x}) = g(\boldsymbol{x})}$ for a. e. $\boldsymbol{x} \in \sigma$.

- **Difficulty**: Non-elliptic principal part.
  Appropriate framework provided by $H^1_\Theta(\Omega) := \left\{ V \in L^2(\Omega) \;/\; {}^t\Theta\, \boldsymbol{\nabla} V \in L^2(\Omega; \mathbb{C}^2) \right\}$.
- **Advantage**: $\mathbb{A}_p$ and ρ_p are periodic in the direction of the interface Σ.

Solution of the augmented problem

① Choose G such that $G(\cdot + \mathbf{e}_2) = G$

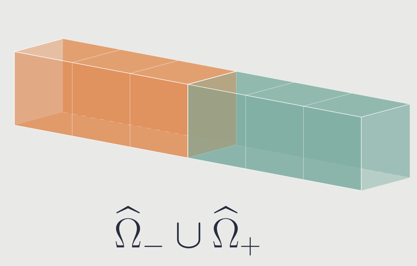
Then $U(\cdot + \mathbf{e}_2) = U(\cdot)$ and $(\mathscr{P}\text{aug})$ reduces to a problem set in the strip $\Omega_+ \cup \Omega_-$ with $\Omega_\pm := \mathbb{R}_\pm \times \mathbb{R} \times (0, 1)$.



$$\left| \begin{array}{l} -\,{}^t\boldsymbol{\nabla}\, \Theta\, \mathbb{A}_p\, {}^t\Theta\, \boldsymbol{\nabla} U - \rho_p\,\omega^2\, U = 0 \quad \text{in} \quad \Omega_+ \cup \Omega_- \\ [\![(\Theta\, \mathbb{A}_p\, {}^t\Theta\, \boldsymbol{\nabla} U) \cdot \mathbf{e}_x]\!]_{x=0} = G \quad \text{on} \quad \{0\} \times \mathbb{R} \times (0, 1) \\ U(\cdot + \mathbf{e}_2) = U(\cdot). \end{array} \right.$$

② Apply the Floquet-Bloch transform in the $\mathbf{e}_1$–direction

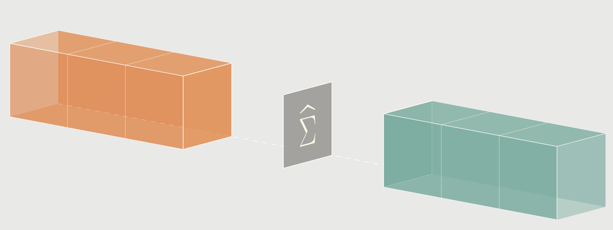
Solve a family of problems set in the waveguide $\widehat{\Omega}_+ \cup \widehat{\Omega}_-$ with $\widehat{\Omega}_\pm := \mathbb{R}_\pm \times (0, 1)^2$, and indexed by the Floquet variable $k \in (-\pi, \pi)$.



$$\left| \begin{array}{l} -\,{}^t\boldsymbol{\nabla}\, \Theta\, \mathbb{A}_p\, {}^t\Theta\, \boldsymbol{\nabla} \widehat{U}_k - \rho_p\,\omega^2\, \widehat{U}_k = 0 \quad \text{in} \quad \widehat{\Omega}_+ \cup \widehat{\Omega}_- \\ [\![(\Theta\, \mathbb{A}_p\, {}^t\Theta\, \boldsymbol{\nabla} \widehat{U}_k) \cdot \mathbf{e}_x]\!]_{\widehat{\Sigma}} = \widehat{G}_k \quad \text{on} \quad \widehat{\Sigma} := \{0\} \times (0, 1)^2 \\ \widehat{U}_k(\cdot + \mathbf{e}_1) = e^{ik}\, \widehat{U}_k(\cdot) \quad \text{and} \quad \widehat{U}_k(\cdot + \mathbf{e}_2) = \widehat{U}_k(\cdot). \end{array} \right.$$

③ Reduction to an interface equation

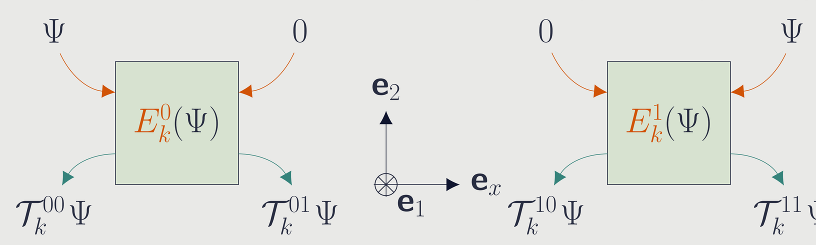
With $\Phi := \widehat{U}_k|_{\widehat{\Sigma}}$, the solution $\widehat{U}_k$ coincides on $\widehat{\Omega}_\pm$ with $\widehat{U}_k^\pm(\Phi)$ which satisfies



$$\left| \begin{array}{l} -\,{}^t\boldsymbol{\nabla}\, \Theta\, \mathbb{A}_p\, {}^t\Theta\, \boldsymbol{\nabla} \widehat{U}_k^\pm(\Phi) - \rho_p\,\omega^2\, \widehat{U}_k^\pm(\Phi) = 0 \quad \text{in} \quad \widehat{\Omega}_\pm \\ \widehat{U}_k^\pm(\Phi) = \Phi \quad \text{on} \quad \widehat{\Sigma} \\ \widehat{U}_k^\pm(\cdot + \mathbf{e}_1) = e^{ik}\, \widehat{U}_k(\cdot) \quad \text{and} \quad \widehat{U}_k^\pm(\cdot + \mathbf{e}_2) = \widehat{U}_k^\pm(\cdot). \end{array} \right.$$

$$\boxed{(\widehat{\Lambda}_k^+ + \widehat{\Lambda}_k^-)\, \Phi = \widehat{G}_k \text{ on } \widehat{\Sigma}} \quad \text{with} \quad \boxed{\widehat{\Lambda}_k^\pm \Phi := \mp \left(\Theta\, \mathbb{A}_p\, {}^t\Theta\, \boldsymbol{\nabla} \widehat{U}_k^\pm(\Phi)\right) \cdot \mathbf{e}_x|_{\widehat{\Sigma}}}$$

- Compute $E_k^0(\Psi)$ and $E_k^1(\Psi)$ and local DtN operators $\mathcal{T}_k^{\ell j}$
  $E_k^0(\Psi)$ and $E_k^1(\Psi)$ satisfy the same PDE and periodicity conditions as $\widehat{U}_k^+(\Phi)$ in $\mathcal{C} := (0, 1)^3$, with Dirichlet conditions on $\{x = 0\}$ and $\{x = 1\}$.

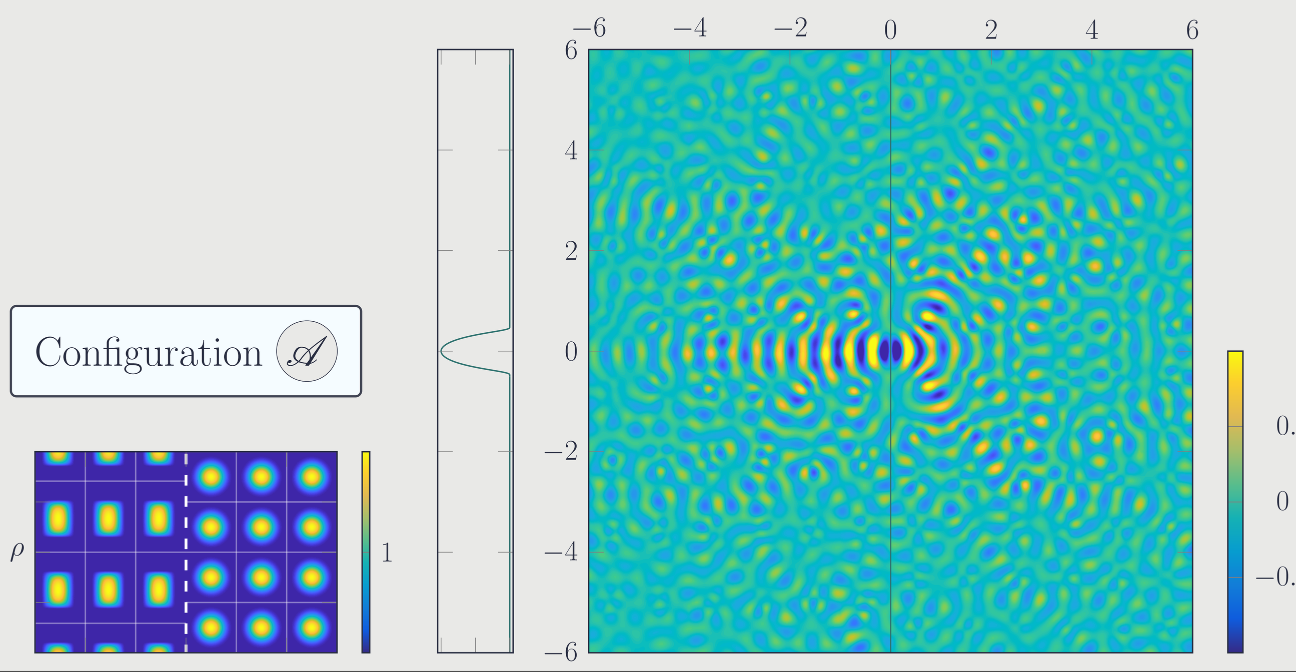


$$\boxed{\mathcal{T}_k^{\ell j}\Psi := (-1)^{j+1} \left(\Theta\, \mathbb{A}_p\, {}^t\Theta\, \boldsymbol{\nabla} E_k^\ell(\Psi)\right) \cdot \mathbf{e}_x|_{x=j}}$$

- Find the propagation operator $\mathcal{P}_k$ which is the unique solution of the Riccati equation $\boxed{\mathcal{T}_k^{10}\, \mathcal{P}_k^2 + (\mathcal{T}_k^{00} + \mathcal{T}_k^{11})\, \mathcal{P}_k + \mathcal{T}_k^{01} = 0}$ with a spectral radius $\boxed{\rho(\mathcal{P}_k) < 1}$
- Finally $\boxed{\widehat{\Lambda}_k^+ = \mathcal{T}_k^{10}\, \mathcal{P}_k + \mathcal{T}_k^{00}}$ and $\boxed{\widehat{U}_k^+(\Phi)(\cdot + n\, \mathbf{e}_x)|_{\mathcal{C}} = E_k^0(\mathcal{P}_k^n \Phi) + E_k^1(\mathcal{P}_k^{n+1}\Phi) \;\forall\, n \in \mathbb{N}}$

[*Fliss, Joly, Li*, 2006] **Dirichlet-to-Neumann method to compute $\widehat{\Lambda}_k^+$** (same for $\widehat{\Lambda}_k^-$)

Numerical results



$$\boxed{\begin{array}{c} \omega = 20 + 0.05\, \mathrm{i} \\ \mathbb{A} = \mathrm{Id} \end{array}}$$

Perspectives

- Extension to $\mathfrak{Im}\,\omega = 0$
- Can this approach be used to compute interface modes?